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Topologically nontrivial sectors of the Maxwell field theory on algebraic curves

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Abstract

In this paper a large family of nondegenerate metrics is derived on general algebraic curves. In this way, it is possible to treat many differential equations arising in quantum mechanics and field theories on Riemann surfaces as differential equations on the complex sphere. The case of the Maxwell field theories on curves with Z_n group of automorphisms is studied in details. These curves are particularly important because they cover the entire moduli space spanned by the Riemann surfaces of genus $g \leq 2$. All the classical solutions of the Maxwell equations are explicitly constructed. Also the examples of the scalar fields and of an electron immersed in a constant magnetic field will be briefly investigated. © 1998 Elsevier Science B.V.

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1. Introduction

Two-dimensional gauge field theories on Riemann surfaces have recently been considered by various authors [1,2]. The abelian case is particularly interesting because of the presence of topologically nontrivial gauge configurations [3], which are relevant in string theories [4], in topological quantum mechanics [5] and in the theory of the quantum Hall effect

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(QHE) [6,7]. These configurations have been derived in [7] for particular metrics and, more recently, in [8] for any metric. In this paper, we consider the abelian gauge field theory, or Maxwell field theory, on algebraic curves [13]. The latter provide an explicit representation of Riemann surfaces as n-sheeted coverings of the complex sphere. Using the formalism of algebraic curves, physically relevant results have already been obtained in string theory [10–12]. In our case, the advantage of working on algebraic curves is that any differential equation defined on a Riemann surface, like those arising for instance in quantum field theory or quantum mechanics, can be rewritten as a differential equation on the sphere. Moreover, at least in the case of algebraic curves with Z_n symmetry group of automorphisms, the moduli are explicitly given by the branch points and enter in the equation of the curve as simple complex parameters.

However, the analytic solution of the physically relevant field or wave equations on algebraic curves remains very difficult. Until now, only the $b-c$ systems have been fully solved using the operator formalism of Refs. [10,11]. The Maxwell field theory is more complicated because it is not conformally invariant and, as a consequence, the multivalued metric tensor is coupled to the fields. Unfortunately, one cannot exploit on algebraic curves the powerful methods of the theory of theta functions [9]. As a matter of fact, the existing formulas of the prime form are too complicated and unexplicit for physical calculations so that it is not possible to construct the topologically nontrivial gauge configurations as in [8]. On the other side, using the so-called $C\theta M$ metric of [7] there is no need of the prime form, but the expression of the canonically normalized differentials is required, which is not known (in [14] these differentials have been constructed up to a theta constant).

To circumvent these difficulties, we restrict ourselves to algebraic curves with Z_n group of automorphisms. This is an important class of curves which includes the hyperelliptic ones and covers the entire moduli space spanned by the Riemann surfaces of genus $g \leq 2$. We show that a particular feature of the Z_n symmetric curves is the existence of a big family of nondegenerate metrics which are singlevalued on $\mathbf{CP}_1$. The latter property greatly simplifies the calculations and allows the derivation of all the classical solutions of the Maxwell field theory. The topologically nontrivial gauge fields obtained here have a relatively simple expression, similar to their analogues on the complex sphere and thus can provide new insights in two-dimensional quantum mechanics and QHE on a manifold. The above metrics are also generalized to arbitrary algebraic curves, but in this case they are no longer single-valued and we are not able to solve the Maxwell equations without introducing the prime form. Nevertheless, it is still possible to write in terms of multi-valued differential operators on the sphere the equations of motion of particles living on Riemann surfaces. Besides the Maxwell equations, concrete examples briefly treated here are the equations of the scalar fields on general algebraic curves and the Hamiltonian of a massive electron immersed in a constant magnetic field.

The material presented in this paper is divided as follows. In Section 2 the Maxwell field theory on Riemann surfaces and algebraic curves is introduced. The gauge fields are decomposed in their exact, coexact and harmonic components using the Hodge decomposition theorem and the harmonic components are explicitly derived. In Section 3 we construct nondegenerate metrics on the Z_n algebraic curves. For some of these metrics the corresponding

Ricci tensor is computed. We check using the Poincaré–Lelong equation [13] that different curvature tensors yield the same Euler characteristics as expected. In Section 4 the topologically nontrivial solutions of the Maxwell equations are derived. We verify that the magnetic fluxes generated by these gauge fields satisfy the Dirac quantization condition. Finally, in the conclusions (Section 5) we discuss the possible applications and generalizations of our results. In particular, it is shown how to generalize the metrics of Section 3 to any affine algebraic curve. Moreover, the equations of motion of the scalar fields on any algebraic curve are treated with some details and the Hamiltonian of a massive electron immersed in a constant magnetic field is explicitly constructed.

2. The Maxwell field theory on algebraic curves

In this paper we consider the Maxwell field theory on a Riemann surface Σ of genus $h > 1$ and with Z_n group of symmetry. The action is given by

$$S_{\text{Maxwell}} = \int\limits_{\Sigma} \mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \frac{\sqrt{g}}{4} F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu, \nu = 1, 2$ and $g_{\mu\nu}$ is a conformally flat metric with determinant g. We choose on Σ complex coordinates $\xi = \xi^1 + \mathrm{i}\xi^2$ and $\bar{\xi} = \xi^1 - \mathrm{i}\xi^2$ so that the components of the field strength ad of the metric become, respectively,

$$F_{\xi\bar{\xi}} = -F_{\bar{\xi}\xi} = -\frac{1}{2\mathrm{i}} F_{12}$$

and

$$g_{\xi\bar{\xi}} = g_{\bar{\xi}\xi} = \tfrac{1}{2}\sqrt{g}, \qquad g_{\xi\xi} = g_{\bar{\xi}\bar{\xi}} = 0.$$

Moreover, the volume form in complex coordinates is given by $\mathrm{d}^2\xi g_{\xi\bar{\xi}} = \mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \sqrt{g}$, where $\mathrm{d}^2\xi \equiv \mathrm{i}\,\mathrm{d}\bar{\xi} \wedge \mathrm{d}\xi$. Accordingly, the classical equations of motion of the Maxwell field theory take the following form:

$$\partial_{\bar{\xi}}[g^{\xi\bar{\xi}}(\partial_{\bar{\xi}} A_\xi - \partial_\xi A_{\bar{\xi}})] = 0, \tag{1}$$

$$\partial_{\bar{\xi}}[g^{\xi\bar{\xi}}(\partial_\xi A_{\bar{\xi}} - \partial_{\bar{\xi}} A_\xi)] = 0 \tag{2}$$

with $g^{\xi\bar{\xi}}$ being the inverse metric: $g^{\xi\bar{\xi}} g_{\xi\bar{\xi}} = 1$.

Let us decompose the gauge fields using the Hodge decomposition

$$A_\xi = \partial_\xi \varphi + \partial_\xi \rho + A_\xi^{\text{har}} + A_\xi^{\text{I}}, \qquad A_{\bar{\xi}} = -\partial_{\bar{\xi}} \varphi + \partial_{\bar{\xi}} \rho + A_{\bar{\xi}}^{\text{har}} + A_{\bar{\xi}}^{\text{I}}. \tag{3}$$

The coexact and exact components are expressed using the two scalar fields φ and ρ, respectively, where φ is purely imaginary, while ρ is purely real. A_ξ^{har} and $A_{\bar{\xi}}^{\text{har}}$ take into account the holomorphic differentials $A^{\mathrm{har}}_{i\xi}$ and $A^{\mathrm{har}}_{i\bar{\xi}}$, $i = 1, \dots, h$, while the A^{I}_{ξ}, $A^{\mathrm{I}}_{\bar{\xi}}$ represent topologically nontrivial gauge configurations corresponding to nonvanishing values of the first Chern class. The former satisfy the relations:

$$\partial_{\bar{\xi}} A^{\mathrm{har}}_{i,\xi} = \partial_{\xi} A^{\mathrm{har}}_{i,\bar{\xi}} = 0,$$

while the latter are built in such a way that

$$F_{\xi\bar{\xi}}\,\mathrm{d}\xi \wedge \mathrm{d}\bar{\xi} = \frac{\mathrm{i}\Phi}{A} g_{\xi\bar{\xi}}\,\mathrm{d}\bar{\xi} \wedge \mathrm{d}\xi,$$

where Φ is a constant representing the total magnetic flux associated with the fields A^{I}_{ξ}, $A^{\mathrm{I}}_{\bar{\xi}}$ and $A = \mathrm{i}\int_{\Sigma} \mathrm{d}^2\xi\, g_{\xi\bar{\xi}}$ denotes the area of the Riemann surface.

At this point, we suppose that Σ has a Z_n group of automorphisms and we represent it explicitly as an algebraic curve determined by the vanishing of Weierstrass polynomials of the following kind:

$$y^n = \prod_{i=1}^{nm}(z - a_i). \tag{4}$$

In (4), z and $\bar{z}$ denote a set of complex variables describing the sphere $\mathbf{CP}_1$ and n, m are integers. In this formalism z can be viewed as a mapping $z : \xi \in \Sigma \to \mathbf{CP}_1$. Here it will be always understood that z is a function of ξ, i.e. $z = z(\xi)$, unless conversely stated. For our purposes, it will also be convenient to regard $\mathbf{CP}_1$ as the compactified complex plane, i.e. $\mathbf{CP}_1 \equiv \mathbf{C} \cup \{\infty\}$. As usual, we cover $\mathbf{CP}_1$ with two open sets U_1 and U_2 containing the points $z = 0$ and $z = \infty$, respectively. The local coordinates z' on U_2 is related to z by the conformal transformation: $z' = 1/z$. The coefficients a_i appearing in (4) are complex parameters denoting the branch points of the curve. It is easy to check that the point at infinity $z = \infty$ is not a branch point. Solving Eq. (4) with respect to y, we obtain a multivalued function $y(z)$, whose branches will be denoted with symbol $y^{(l)}(z)$, $l = 0, \dots, n-1$. A generic tensor $T(z, \bar{z})$ on the algebraic curve can be multivalued on the complex sphere due to its dependence on $y^{(l)}(z)$. Its branches will be denoted as follows:

$$T^{(l)}(x, \bar{z}) \equiv T(z, \bar{z}; y^{(l)}(z), \bar{y}^{(l)}(\bar{z})). \tag{5}$$

Fig. 1. A possible set of branch cuts on the complex sphere for the Z_n symmetric algebry curves. The cuts appear symmetrically on the sheets composing the curve.

To construct a system of branch lines which is consistent with the multi-valuedness of the function $y(z)$, we group the branch points in m different sets $I_i = \{a_{(i-1)n+1}, \ldots, a_{in}\}$, with $i = 1, \ldots, m$. The branch points in a given set I_i are connected together by branch lines as shown in Fig. 1. As a convention, going around a branch point clockwise (counterclockwise) on the jth sheet along a small circle surrounding the point, one encounters the $(j+1)$th $((j-1)$th) sheet when crossing a branch line, where $j = 0, \ldots, n-1 \mod n$.

The genus of the Riemann surface (4) is

$$h = 1 - n + \tfrac{1}{2}(nm(n-1)). \tag{6}$$

In order to describe the topologically nontrivial solutions of the Maxwell field theory on the algebraic curves (4) explicitly, the following divisors are necessary: [2]

$$\begin{aligned} [\,\mathrm{d}z] &= (n-1)\sum_{p=1}^{nm} a_p - 2\sum_{j=0}^{n-1} \infty_j, \\ [y] &= \sum_{p-1}^{nm} a_p - m\sum_{j=0}^{n-1} \infty_j. \end{aligned} \tag{7}$$

In (7) the symbol ∞_j denotes the projection of the point $z = \infty$ on the jth sheet. Exploiting the above divisors, it is easy to see that the holomorphic differentials $A^{\mathrm{har}}_{i,\xi}$ and $A^{\mathrm{har}}_{i,\bar{\xi}}$ correspond to linear combinations of the following holomorphic differentials:

$$\Omega_{k,j}\,\mathrm{d}z = \frac{z^{j-1}}{y^{-k+n-1}}\,\mathrm{d}z, \tag{8}$$

where for $m > 1$:

$$j = 1, \ldots, (n-1)m - km - 1, \qquad k = 0, \ldots, n-2,$$

and for $m = 1$:

$$j = 1, \ldots, n-k-2, \qquad k = 0, \ldots, n-3.$$

The calculation of the topologically nontrivial solutions A^{l}_{ξ} will be the subject of Section 3.

3. Metric tensors on algebraic curves

For the calculation of the topologically nontrivial configurations the knowledge of at least one nondegenerate metric tensor on Σ is necessary. A simple class of conformally flat metrics is provided by tensors of the kind:

$$g_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z} = \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(y\bar{y})^{n-1}}[1 + f(z,y)\overline{f(z,y)}]^{\alpha}, \tag{9}$$

[2] These divisors can be computed using the methods of Refs. [11,12].

where $f(z, y)$ is a rational function of z and y and $\overline{f(z, y)}$ its complex conjugate. Clearly, the parameter α can be any real number without introducing further branches on $\mathbf{CP}_1$. For $\alpha = ((n-1)m - 2)/m$ and $f(z, y) = y(z)$, we have the nondegenerate metric

$$g_{2,z\bar{z}}\, dz\, d\bar{z} = \frac{dz\, d\bar{z}}{(y\bar{y})^{n-1}}[1 + y\bar{y}]^{\alpha}. \tag{10}$$

The curvature scalar R_2 and the curvature two-form $R_{2,z\bar{z}}$ [15] corresponding to the above conformally flat metric are, respectively, given by

$$R_2 = -\frac{\alpha}{2}, \qquad R_{2,z\bar{z}} = -2\alpha \frac{\partial_z y \partial_{\bar{z}} \bar{y}}{[1 + y\bar{y}]^2}. \tag{11}$$

Another interesting example of nondegenerate metric is provided by

$$g_{3,z\bar{z}}\, dz\, d\bar{z} = \frac{dz\, d\bar{z}}{(y\bar{y})^{n-1}}[1 + z\bar{z}]^{\beta} \tag{12}$$

for $\beta = (n-1)m - 2$. The curvature scalar and the curvature two-form corresponding to this metric have a very simple form:

$$R_3 = -\frac{\beta}{2}, \qquad R_{3,z\bar{z}} = -\frac{2\beta}{(1 + z\bar{z})^2}. \tag{13}$$

Now we verify that the above metric tensors yield the exact Euler characteristic χ. On a Riemann surface Σ of genus h, represented as an algebraic curve, χ is defined as follows:

$$\chi = \frac{1}{4\pi} \int_{\Sigma} d^2 z(\xi) g_{z\bar{z}} R \equiv 2 - 2h, \tag{14}$$

where R is the curvature scalar. We recall at this point that the integral of a density $L^{(l)}_{z\bar{z}}(z, \bar{z})$ on an algebraic curve Σ can be rewritten as a discrete sum of integrals over $\mathbf{CP}_1$:

$$\int_{\Sigma} d^2 z(\xi) L^{(l)}_{z\bar{z}}(z(\xi), \bar{z}(\bar{\xi})) = \sum_{t=0}^{n-1} \int_{\mathbf{CP}_1} d^2 z L^{(l)}_{z\bar{z}}(z, \bar{z}). \tag{15}$$

Eq. (15) can be rigorously proved by means of the Poicaré–Lelong equation [13]. Putting $L^{(l)}_{z,\bar{z}}(z\bar{z}) = (1/4\pi) g_{z\bar{z}} R_3$ in Eq. (15) we obtain for the Euler characteristic:

$$\chi = -\frac{n}{4\pi} \int_{\mathbf{CP}_1} d^2 z \frac{2\beta}{(1 + z\bar{z})^2} = -\beta n. \tag{16}$$

To derive the right-hand side of (16) we have used the fact that the integral over $\mathbf{CP}_1$ is proportional to the Euler characteristic of the complex sphere:

$$\chi_{\mathbf{CP}_1} = \frac{1}{\pi} \int_{\mathbf{CP}_1} \frac{d^2 z}{(1 + z\bar{z})^2} = 2. \tag{17}$$

Therefore, Eq. (16) yields $\chi = -n\beta = 2n - nm(n-1)$. Comparing this value of χ with Eq. (6), which gives the genus of Σ in terms of n and m, it is easy to see that $\chi = 2 - 2g$ as expected. The computation of the Euler characteristic starting from the Ricci tensor $R_{2,z\bar{z}}$ of Eq. (11) can be performed in an analogous way considering y as the independent variable in Eq. (4), so that $z(y)$ becomes a multi-valued function with nm branches over $\mathbf{CP}_1$.

Let us finally notice that we can construct other metric tensors on an algebraic curve which are not of the form (9). For instance the tensor

$$\tilde{g}_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z} = \frac{\mathrm{e}^{(y\bar{y})^{n-1}}}{\mathrm{e}^{(y\bar{y})^{n-1}-1}}[1+z\bar{z}]^{2/m}\,\mathrm{d}z\,\mathrm{d}\bar{z} \tag{18}$$

yields a nondegenerate metric, as it is easy to check exploiting the divisors (7). Moreover, all the metric tensors given above are characterized by the fact that they are not multi-valued on $\mathbf{CP}_1$. In fact, they depend on y only through the modulus of this function, which is branch independent. Nondegenerate metrics which are also multi-valued can be for instance obtained multiplying Eqs. (10), (12) and (18) by integer powers of the factor $y/y+\bar{y}/y$ or by considering more complicated forms of the functions $f(z,y)$ in Eq. (9)

4. Solitonic sectors of the Maxwell field theory

At this point, we are ready to derive the fields A^{I}_{ξ} and $A^{\mathrm{I}}_{\bar{\xi}}$. On the algebraic curve, this is equivalent to solve the equation:

$$F_{z\bar{z}}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} = \frac{\mathrm{i}\Phi}{A}g_{z\bar{z}}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}. \tag{19}$$

The difficulty of computing A^{I}_{ξ} and $A^{\mathrm{I}}_{\bar{\xi}}$ explicitly strongly depends on the choice of the metric $g_{z\bar{z}}$. In the formalism of theta functions, Eq. (19) can be easily solved in the canonical θ-metric (CθM) of [7]. Unfortunately, it is not possible to construct this metric on algebraic curves, since the expression of the period matrix in terms of the branch points is not known. To simplify our calculations, we choose single-valued metrics on $\mathbf{CP}_1$ as those in Eqs. (10), (12) and (18). To solve Eq. (19), we first define the follwing Green function:

$$G(z,w) = -\frac{1}{4\pi n}\log\left[\frac{|z-w|^2}{(1+z\bar{z})(1+w\bar{w})}\right].$$

Denoting with $J(z,\bar{z})$ an external single-valued scalar current on $\mathbf{CP}_1$, it is possible to show by means of Eq. (15) that, if the metric is single-valued, $G(z,w)$ satisfies the following relation:

$$\begin{aligned}\partial_z\partial_{\bar{z}}\int\limits_{\Sigma}&\mathrm{d}^2w(\xi')G(w(\xi'),z(\xi))J(w(\xi'),\bar{w}(\bar{\xi}'))g_{w\bar{w}}\\ &= -\frac{J(z,\bar{z})}{2}g_{z\bar{z}}+\frac{\gamma_{z\bar{z}}}{4\pi n}\int\limits_{\Sigma}\mathrm{d}^2w\,J(w,\bar{w})g_{w\bar{w}},\end{aligned} \tag{20}$$

where $\gamma_{z\bar{z}}\,\mathrm{d}z\,\mathrm{d}\bar{z} = \mathrm{d}z\,\mathrm{d}\bar{z}/(1+z\bar{z})^2$.

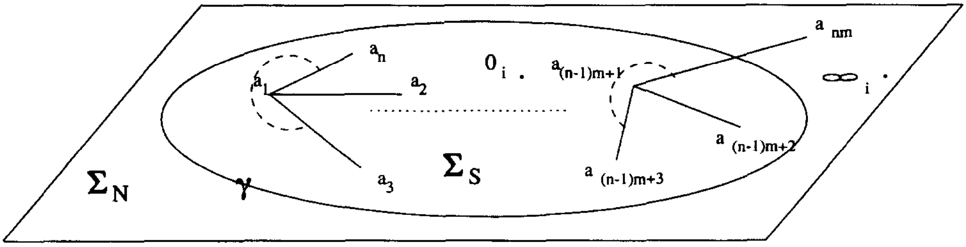

Fig. 2. A possible covering of the curve Σ in two sets Σ_N and Σ_S. Only the part of the contour γ which lies on the ith sheet is showed.

Besides the Green function $G(z, w)$, we also introduce a gauge field A_α^{sph}, $\alpha = z, \bar{z}$, defined in this way:

$$A_z^{\mathrm{sph}} = -\frac{A}{4\pi n}\partial_z \log(1 + z\bar{z}), \quad A_{\bar{z}}^{\mathrm{sph}} = \frac{A}{4\pi n}\partial_{\bar{z}} \log(1 + z\bar{z}). \tag{21}$$

It is easy to check that the relation

$$\partial_z A_{\bar{z}}^{\mathrm{sph}} - \partial_{\bar{z}} A_z^{\mathrm{sph}} = \frac{A\gamma_{z\bar{z}}}{4\pi n} \tag{22}$$

is satisfied over all the algebraic curve Σ apart from the points $\infty_0, \ldots, \infty_{n-1}$. At those points, in fact, a δ-function concentrated in $n = \infty$ appears in the right-hand side of (22). The problem of the appearance of δ functions is solved here as in the case of the Wu–Yang monopoles on the sphere by splitting the algebraic curve into two sets Σ_S and Σ_N. The former should contain all the projections of the point $z = 0$ but not those of the point $z = \infty$, while for Σ_N the converse is true. Of course, there is a great arbitrainess in choosing these sets. To fix the ideas, we will define them as the two sets obtained by cutting the algebraic curve along the contour γ shown in Fig. 2. Thus Σ_S encloses the projections $0_0 \ldots, 0_{n-1}$ of the point $z = 0$ and all the branch points apart from a_{nm}. Consequently, Σ_N includes the points $\infty_0, \ldots, \infty_{n-1}$ and the branch point a_{nm}. The contour of Fig. 2 is valid also in the case in which $z = 0$ is a branch point. In fact we can always put $a_1 = 0$, without any loss of generality. Σ_S and Σ_N are not isomorphic to $\mathbf{C}$, but this way of covering the algebraic curve will be sufficient for our purposes as we will show below. Indeed, let us write the solution of Eq. (19) on Σ_S:

$$A_{\bar{z}}^S = \frac{\mathrm{i}\Phi}{A}\left[\int_\Sigma \mathrm{d}^2 w \partial_z G(z, w) g_{w\bar{w}} + A_{\bar{z}}^{\mathrm{sph}}\right], \tag{23}$$

$$A_z^S = \frac{\mathrm{i}\Phi}{A}\left[-\int_\Sigma \mathrm{d}^2 w \partial_{\bar{z}} G(z, w) g_{w\bar{w}} + A_z^{\mathrm{sph}}\right]. \tag{24}$$

Exploiting Eqs. (20) and (22), it turns out that

$$(\partial_z A_{\bar{z}}^S - \partial_{\bar{z}} A_z^S)\, \mathrm{d}z \wedge \mathrm{d}\bar{z} = \frac{\mathrm{i}\Phi}{A} g_{z\bar{z}}\, \mathrm{d}\bar{z} \wedge \mathrm{d}z \tag{25}$$

as desired. Analogous expressions of the solutions of Eq. (19) can be written on Σ_N in coordinates $z' = 1/z$ and $\bar{z}' = 1/\bar{z}$:

$$A^N_{z'} = \frac{\mathrm{i}\Phi}{A}\left[\int_\Sigma \mathrm{d}^2w' \partial_{z'} G(z', w') g_{w'\bar{w}'} + \tilde{A}^{\mathrm{sph}}_{z'}\right], \tag{26}$$

$$A^N_{\bar{z}'} = \frac{i\Phi}{A}\left[-\int_\Sigma \mathrm{d}^2w' \partial_{\bar{z}'} G(z', w') g_{w'\bar{w}'} + \tilde{A}^{\mathrm{sph}}_{\bar{z}'}\right], \tag{27}$$

where

$$\tilde{A}^{\mathrm{sph}}_{z'} = -\frac{A}{4\pi n}\partial_{z'} \log(1 + z'\bar{z}'), \qquad \tilde{A}^{\mathrm{sph}}_{\bar{z}'} = \frac{A}{4\pi n}\partial_{\bar{z}'} \log(1 + z'\bar{z}'). \tag{28}$$

As for the fields A^S_z and $A^S_{\bar{z}}$, it is easy to prove that the following relation is satisfied:

$$(\partial_{z'} A^N_{\bar{z}'} - \partial_{\bar{z}'} A^N_{z'}) = \frac{\mathrm{i}\Phi}{A} g_{z'\bar{z}'} \,\mathrm{d}\bar{z}' \wedge \mathrm{d}z' \tag{29}$$

Eqs. (23), (24) and (26),(27) show the reasons for which we only need two sets to cover the Riemann surface Σ. First of all, the gauge fields A^N_α and A^S_α with $\alpha = z, \bar{z}$, are single-valued on Σ. Secondly, they are everywhere not singular, apart from the projections on the algebraic curve of the points $z = 0, \infty$. As a consequence, the behaviour of the gauge fields $A^{N,S}_\alpha$ is not affected by the presence of the branch points and the splitting of Σ into two sets Σ_N and Σ_S is justified.

To be consistent, both fields A^N_α and A^S_α should describe the same magnetic field. Indeed, far from the points $z = 0$ and $z = \infty$, where it is possible to use both coordinates $z(\xi)$ and $z'(\xi)$, one can see that the fields $A^{N,S}_\alpha$ are related by a gauge transformation:

$$A^S_{z'} \,\mathrm{d}z' = A^N_{z'} \,\mathrm{d}z' + \partial_{z'} \Lambda \,\mathrm{d}z', \qquad A^S_{\bar{z}'} \,\mathrm{d}\bar{z}' = A^N_{\bar{z}'} \,\mathrm{d}\bar{z}' + \partial_{\bar{z}'} \Lambda \,\mathrm{d}\bar{z}', \tag{30}$$

where, using local polar coordinates $z' = (1/\rho')\mathrm{e}^{-\mathrm{i}\theta'}$, we have that $\Lambda = \Phi\theta'/2\pi n$. The transformations (30) consists in a $U(1)$ gauge transformation with group element $U(z', \bar{z}')$ given by

$$U = \mathrm{e}^{\mathrm{i}\Lambda} = \mathrm{e}^{\mathrm{i}(\Phi\theta'/2\pi n)}. \tag{31}$$

Since the fields A^N_α and A^S_α differ by an exact differential, the corresponding field strength $F_{z\bar{z}}$ is globally defined on Σ and we can compute the total magnetic flux associated to the gauge field configurations of Eqs. (23), (24) and (26), (27). Using Eqs. (25) and (29) we have

$$\int_\Sigma \mathrm{d}^2z F^{\mathrm{I}}_{z\bar{z}} = \int_{\Sigma_S} \mathrm{d}^2z F^S_{z\bar{z}} + \int_{\Sigma_N} \mathrm{d}^2z' F^N_{z'\bar{z}'} = \Phi$$

as desired, where $F^{\mathrm{I}}_{z\bar{z}} = \partial_z A^{\mathrm{I}}_{\bar{z}} - \partial_{\bar{z}} A^{\mathrm{I}}_z$ and $\mathrm{I} = N, S$ on $\Sigma_{N,S}$. Still we have to check that the group element U defined in Eq. (31) is single-valued when transported along the path γ

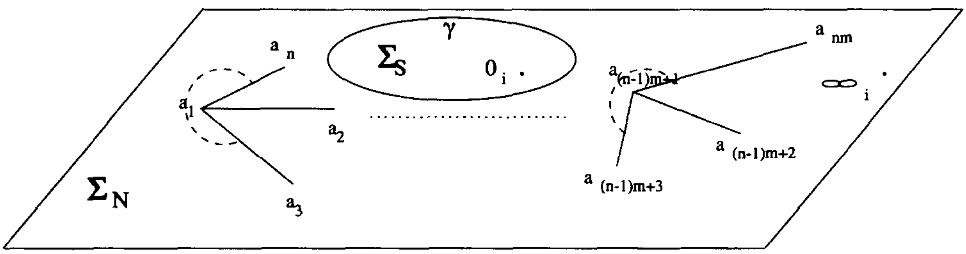

Fig. 3. An alternative form of the two sets Σ_N and Σ_S. Σ_S is disconnected in n pieces lying on the different sheets. In the figure only the piece belonging to the ith sheet has been given.

and the nontrivial homology cycles of Σ. When U is transported N times along γ, the angle θ' undergoes the shift: $\theta' \to \theta' + 2\pi nN$. The factor n is due to the fact that the contour γ encircles all the projections of the point $z = \infty$ on the n different sheets composing the algebraic curve. Thus, in order to ensure the single-valuedness of U, the following condition on the total flux Φ should be imposed: $\Phi = 2\pi k$, with $k = 0, \pm 1, \pm 2, \ldots$ As a consequence, the solutions of Eq. (19) provided by the gauge field configurations in (23), (24) and (26), (27) satisfy the relation:

$$2\pi k = \int_{\Sigma} \mathrm{d}^2 z F_{z\bar{z}} \tag{32}$$

for integer values of k. As expected, Eq. (32) is exactly the Dirac quantization condition of the magnetic flux. This result does not depend on the form of the contour γ. In fact, a curve which encircles all the projections of the points $z = 0$ and $z = \infty$ has either to cross at least n branch lines as in Fig. 2 or to be of the form given in Fig. 3. In both cases, the total shift in the angle θ' has an n factor in front which does not allow for fractional values of k in Eq. (32). Finally, the gauge fields defined in Eqs. (23), (24) and (26), (27) are single-valued on $\mathbf{CP}_1$, so that no problem arises when they are transported along the homology cycles.

5. Conclusions

In Eqs. (8) and (23)–(27) we have computed all the nontrivial solutions of the Maxwell equations on a Z_n symmetric algebraic curve. The gauge fields configurations (23), (24) and (26), (27) have a simple form, very similar to that of their counterparts on the complex sphere. Using the above gauge potentials it is possible to write explicitly the Hamiltonian H of an electron of mass m in the presence of a constant magnetic field B perpendicular to Σ. Remembering that $B = \Phi/A$, we have on $\Sigma_{N,S}$:

$$H = [2g^{z\bar{z}}/m](P_z - A_z^{N,S})(P_{\bar{z}} - A_{\bar{z}}^{N,S}) + [\Phi/2mA] \tag{33}$$

with $P_\alpha = -\mathrm{i}\partial_\alpha, \alpha = z, \bar{z}$. $g^{z\bar{z}}$ is the inverse of one of the single-valued metrics given in Section 3. The degenerate ground state of H is given by

$$\Psi_{N,S} = \mathrm{e}^{-\int_{\bar{z}_0}^{\bar{z}} A_{\bar{z}}^{N,S}\, \mathrm{d}\bar{z}} \Psi_0,$$

where Ψ_0 satisfies the relation: $\partial_z \Psi_0 = 0$. due to the Dirac quantization condition of the magnetic flux and the single-valuedness of $A_\alpha^{N,S}$, $\Psi_{N,S}$ is a well-defined quantum mechanical state. In particular, it is periodic along the homology cycles. The metrics given in Section 3 can be easily extended to any algebraic curve with Weierstrass polynomial:

$$F(z, y) = \sum_{i=0}^{n} y^i P_i(z), \tag{34}$$

where the $P_i(z)$ are polynomials in z. In this case, the metrics (9) takes the form:

$$g_{z\bar{z}}\, \mathrm{d}z\, \mathrm{d}\bar{z} \frac{\mathrm{d}z\, \mathrm{d}\bar{z}}{|F_y|^2} (1 + f(z, y)\overline{f(z, y)})^\alpha$$

with $F_y(z, y) = \partial_y F(z, y)$. The values of the parameter α depend on the form of the polynomial (34) and of the function $f(z, y)$. To determine α, one has to derive the divisors of $\mathrm{d}z$, y and F_y as in Eq. (7). For a large class of algebraic curves, such divisors can be found in [11]. Analogously, the metric (18) becomes on a general algebraic curve:

$$\tilde{g}_{z\bar{z}}\, \mathrm{d}z\, \mathrm{d}\bar{z} = \frac{\mathrm{e}^{|F_y|^2}}{\mathrm{e}^{|F_y|^2} - 1} [1 + z\bar{z}]^\beta\, \mathrm{d}z\, \mathrm{d}\bar{z} \tag{35}$$

for suitable values of β. In this way the Lagrangians of many field theories can be explicitly written on algebraic curves. For instance, let us write the action for the scalar fields $\varphi(z, \bar{z}; y^{\tilde{l}}(z), \overline{y^{\tilde{l}}(z)})$ with mass μ:

$$\begin{aligned} S = \sum_{\tilde{l}=0}^{n-1} \int_{\mathbf{CP}_1} & \mathrm{d}^2 z [\tfrac{1}{2}(\mathrm{d}_z \varphi^{(\tilde{l})}\, \mathrm{d}_{\bar{z}} \varphi^{(\tilde{l})} + \mu^2 g_{z\bar{z}}^{(\tilde{l})} (\varphi^{(\tilde{l})})^2 + \lambda_1 R_{z\bar{z}}^{(\tilde{l})} (\varphi^{(\tilde{l})})^2) \\ & + \lambda_2 g_{z\bar{z}}^{(\tilde{l})} R^{(\tilde{l})} \varphi^{(\tilde{l})}]. \end{aligned} \tag{36}$$

Here λ_1, λ_2 represent real parameters and d_z and $\mathrm{d}_{\bar{z}}$ are total derivatives with respect to the variables z and $\bar{z}$. Total derivatives are used to remember that the fields φ depend on $z, \bar{z}$ also through the functions $y(z), \overline{y(z)}$. Deriving the action (36) with respect to the field φ in a given branch l, we find the equation of motion of the scalar fields:

$$-\mathrm{d}_z\, \mathrm{d}_{\bar{z}} \varphi^{(l)} + (\mu^2 g_{z\bar{z}}^{(l)} + \lambda_1 R_{z\bar{z}}^{(l)}) \varphi^{(l)} + \lambda_2 R_{z\bar{z}}^{(l)} = 0. \tag{37}$$

Locally and far from the branch points, it is possible to solve (37) with the standard methods of the theory of partial differential equations on the complex plane. Any local solution derived in this way is in general multi-valued and needs to be analytically continued in order to extend it over the whole algebraic curve. Despite of the difficulties that may arise in the analytic continuation, the possibility of transforming differential equations on a

Riemann surface in differential equations on the sphere is remarkable. Moreover, numerical calculations are allowed due to the explicitness which is intrinsic in the representation of Riemann surfaces in terms n-sheeted coverings of the complex sphere.

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